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Best Approximation, Invariant Measures, and Fixed Points

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We show that M. A. Al-Thagafi's recent generalization of Habiniak and Singh's results on fixed point's best approximation can be extended further to weakly (and weakly*) compact sets. © 2000 Academic Press

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Let $(X, \|\cdot\|)$ be a Banach space and M a closed subset. Without loss of generality we will assume that the zero vector 0 belongs to M. Given a point $p \in X$ by $\delta(p, M)$ we denote the distance from p to M, i.e., $\delta(p, M) = \inf\{\|z-p\| : z \in M\}$. Following notations from [2], $B_M(p)$ stands for $\{z \in M : \|z-p\| = \delta(p, M)\}$, the set of the best M-approximations. We also introduce $M_p = \{z \in M : \|z\| \leqslant 2\|p\|\}$. Clearly $B_M(p) = \{z \in M : \|z-p\| = \delta(p, M)\} \subseteq M_p$ as $0 \in M$.

A mapping φ: $M \to M$ is called *nonexpansive* if $\|\varphi(x) - \varphi(y)\| \leqslant \|x-y\|$ holds for all $x, y \in M$. Given an associative semigroup S, we say that a family $\mathfrak{S} = \{T_s : s \in S\}$ of nonexpansive mappings defined on a common domain $M \subseteq X$ is an *antirepresentation* of S if for each pair $s_1, s_2 \in S$ we have $T_{s_1} \circ T_{s_2} = T_{s_2 s_1}$. A point $x \in M$ is called a *common fixed point of* $\mathfrak{S}$ (we skip "*of* $\mathfrak{S}$" if the context is clear) if $T_s x = x$ holds for every $s \in S$. The set of all common fixed points of $\mathfrak{S}$ belonging to M will be denoted by $F(S, M)$. A subset $K \subseteq M$ is called $\mathfrak{S}$ *invariant* if $T_s(K) \subseteq K$ holds for

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every s. If the set K does not contain any $\mathfrak{S}$ invariant and closed subsets, other than K and the empty set, we will say that K is *minimal*.

A question when $(M, \mathfrak{S})$ has a common fixed point has been addressed in many articles (see [2, 4–6, 8–10]). It is well known that if $\mathfrak{S}$ is commutative and M is convex and norm compact, then a common fixed point does exist. This follows from the Schauder fixed point theorem. If M is a convex, closed, and bounded subset of a Hilbert space (or a uniformly convex Banach space) and $S = \mathbb{N}$ is the semigroup of natural numbers, then a common fixed point exists by [3]. However [1] provides an example of a weakly compact set $M \subseteq L^1$ and a nonexpansive isometry $\varphi: M \to M$ which is fixed point free. The main goal of this paper is to provide sufficient conditions guaranteeing that the best approximation is achieved in the domain of common fixed points, i.e., $F(S) \cap B_M(p) = F(S, B_M(p)) \neq \emptyset$. This idea was investigated by Brosowski [5], Smoluk [10], Habiniak [8], and others. Their efforts were recently summarized in [2] by M. A. Al-Thagafi. Namely it has been proved that

THEOREM 1. *Let X be a normed linear space and I and T be self-maps of X with a common fixed point p of I and T. Let $0 \in M$ be closed and convex in X and $T(M_p) \subseteq I(M) \subseteq M$. Suppose that I is linear and nonexpansive on M_p, $\|Ix - p\| = \|x - p\|$ for all $x \in M$, I and T commute on M_p, T is* I-nonexpansive *on $M_p \cup \{p\}$, and on of the following two conditions is satisfied*:

(a) $\overline{I(M_p)}^{\|\cdot\|}$ *is norm compact*

(b) $\overline{T(M_p)}^{\|\cdot\|}$ *is norm compact and T is linear on M_p.*

Then:

(i) $B_M(p)$ *is nonempty, closed, and convex*;

(ii) $T(B_M(p)) \subseteq I(B_M(p)) \subseteq B_M(p)$, *and*

(iii) *I and T have a common fixed point in $B_M(p)$.*

In the end of the paper [2] the author raised the question whether instead of (a) or (b) we might simply assume that $\overline{T(M_p)}^{\|\cdot\|}$ is norm compact. We answer this question in the affirmative. We generalize this theorem removing redundant restrictions on I and T. We discuss its versions for the weak (and weak*) topology as well.

Remark 1. In the above mentioned Theorem 1, the assumptions that I and T are linear on M_p mean that these mappings are simply the restrictions of linear maps. In our last result, Theorem 5, we will note that it is enough to assume that they are affine. This clarifies the situation as linearity of I and T requires that M_p should be at least a vector subspace.

In the following we will apply a fixed point theorem which was recently proved by one of the authors. In order to make the notations and concepts clear, let us recall a few paragraph from [4]. We shall assume that besides the norm $\|\cdot\|$ our space X is endowed with a Hausdorff topology $\mathfrak{T}$, weaker than (or equivalent to) the norm topology. A standard example of such a triple $(X, \|\cdot\|, \mathfrak{T})$ is when $\mathfrak{T}$ is the corresponding weak topology on X or the weak* topology if X is a dual Banach space.

Now let us assume that the semigroup S is equipped with a Hausdorff topology such that for each fixed $a \in S$ the mappings $s \to sa$ and $s \to sa$ from S to S are continuous (i.e., S is a semitopological semigroup). The antirepresentation $\mathfrak{S}$ is said to be $\mathfrak{T}$ *continuous* if the mapping

$$S \times M \ni (s, x) \mapsto T_s(x) \in M$$

is jointly continuous when M has the $\mathfrak{T}$ topology. We note that if $\mathfrak{T}$ restricted to M is the norm topology and S has the discrete topology, then any nonexpansive antirepresentation of S on M is jointly continuous.

The Banach algebra of all bounded real valued functions on S endowed with the supremum norm $\|\cdot\|_{\sup}$ is denoted by $C_b(S)$. Given $h \in C_b(S)$ and $a \in S$ we define $h_a(t) = h(ta)$, which obviously belongs to $C_b(S)$ and the operation $h \to h_a$ is a linear contraction on $C_b(S)$. We say that $h \in C_b(S)$ is right uniformly continuous if the mapping

$$S \ni a \mapsto h_a \in (C_b(S), \|\cdot\|_{\sup})$$

is continuous, and by $RUC(S)$ we denote the Banach subalgebra of all such functions h. A linear functional λ on $RUC(S)$ is called a mean if $\lambda(h) \geqslant 0$ for all nonnegative h and $\lambda(\mathbf{1}) = 1$. A mean λ is said to be *right invariant* if $\lambda(h_a) = \lambda(h)$ holds for all $h \in RUC(S)$ and $a \in S$. The semigroup S admitting right invariant means are called *RUC right amenable*.

The reader is referred to [7] for more details and information concerning amenability. We remark that regardless of the existing topology on S (one can even take the discrete one), all commutative semigroups have means which are both left and right invariant (S is amenable).

A probability measure μ on $(M, \mathscr{B}_{\mathfrak{T}, M})$, where $\mathscr{B}_{\mathfrak{T}, M}$ denotes the Borel σ algebra generated by the topology $\mathfrak{T}$ restricted to M, is said to be $\mathfrak{S}$-invariant if for every $A \in \mathscr{B}_{\mathfrak{T}, M}$ and $s \in S$ one has $\mu(T_s^{-1}(A)) = \mu(A)$. In particular μ defines an invariant mean on the subspace of $C_b(S)$ spanned by the functions of the form $S \ni s \to h(T_s(x))$, where h is a bounded and $\mathfrak{T}$ continuous function on M and $x \in M$ is fixed but arbitrary. The family of all $\mathfrak{S}$-invariant probability measures concentrated on a set $K \subseteq M$ is denoted by $P(S, K)$. The topological support (if it exists) of μ is the smallest norm closed set of full measure μ and it is denoted by $\mathrm{supp}(\mu)$.

It was recently proved (compare Lemma 1 and Theorem 1 in [4]) that

LEMMA 1. (α) *Let* M *be a* $\mathfrak{T}$*-closed subset of a Banach space* $(X, \|\cdot\|, \mathfrak{T})$ *and* $\mathfrak{S} = \{T_s : s \in S\}$ *be a nonexpansive* $\mathfrak{T}$ *continuous antirepresentation of* S *on* M, *and let* S *be RUC right amenable. If* $\mathcal{O}_{\mathfrak{T}}(x) = \overline{\{T_s(x): s \in S\}}^{\mathfrak{T}}$ *is* $\mathfrak{T}$ *compact, then it carries a* $\mathfrak{S}$ *invariant probability measure* μ.

(β) *If, moreover,* $\mathcal{O}_{\mathfrak{T}}(x)$ *is norm separable and for each* $y \in M$ *the function* $M \ni z \mapsto \|z - y\| \in \mathbb{R}$ *is* $\mathfrak{T}$ *lower semicontinuous, then for every element* y *from* supp(μ) *the orbit* $\mathcal{O}_{\|\cdot\|}(y) = \overline{\{T_s(y): s \in S\}}^{\|\cdot\|}$ *is norm compact and minimal,* $\operatorname{supp}(\mu)$ *is* $\mathfrak{S}$ *invariant, and* $\{T_s|_{\operatorname{supp}(\mu)}: s \in S\}$ *is a collection of invertible isometries on* $\operatorname{supp}(\mu)$.

(γ) *If, in addition,* M *is convex and* $\mathfrak{T}$ *compact, then* $F(\mathcal{S}, M) \neq \emptyset$.

Remark 2. The assumption that functions $z \mapsto \|z - y\|$ are $\mathfrak{T}$ lower semicontinuous implies that $\mathcal{B}_{\mathfrak{T}, M} = \mathcal{B}_{\|\cdot\|, M}$. This is naturally satisfied if $\mathfrak{T}$ is the weak or weak* topology.

Applying the above result we immediately have:

THEOREM 2. *Let* M *be a closed, convex subset of a Banach space* $(X, \|\cdot\|, \mathfrak{T})$, $p \notin M$ *be arbitrary, and* $\mathfrak{S} = \{T_s : s \in S\}$ *be a nonexpansive and* $\mathfrak{T}$ *continuous antirepresentation of* S *on* $M \cup \{p\}$. *Suppose that* $M \ni z \mapsto \|z - y\|$ *is* $\mathfrak{T}$ *lower semicontinuous for every fixed* $y \in M \cup \{p\}$ *and* S *is RUC right amenable. If* p *is a common fixed point and for some* $r > \delta(p, M)$ *the set* $C_r = \{z \in M : \|z - p\| \leqslant r\}$ *is* $\mathfrak{T}$ *compact, then*

(i) $B_M(p)$ *is nonempty and convex.*

If, moreover, C_r *is norm separable, then*

(ii) $F(S, B_M(p)) \neq \emptyset$.

Proof. The convex and nonempty sets C_r decrease when $r \searrow \delta(p, M)$ and they are eventually $\mathfrak{T}$ compact. Therefore

$$\emptyset \neq B_M(p) = \bigcap_{r > \delta(p, M)} C_r = C_{\delta(p, M)}$$

is $\mathfrak{T}$ compact and convex. Property (ii) follows easily from (γ) because $B_M(p)$ is separable, convex, and $\mathfrak{T}$ compact. ∎

Remark 3. It has been noticed in [4] that if every finitely generated subsemigroup of S is right amenable (this holds if S is commutative) then the separability assumption imposed on C_r in Theorem 2 is redundant if $\mathfrak{T}$ is the weak topology.

Now we are in a position to approach the question raised in [2]. We start with:

THEOREM 3. *Let M be a $\mathfrak{T}$ closed subset of a Banach space $(X, \|\cdot\|, \mathfrak{T})$, let the family $\mathfrak{S} = \{T_s : s \in S\}$ be a nonexpansive and $\mathfrak{T}$ continuous antirepresentation of S on M, and let the functions $M \ni z \mapsto \|z - y\|$ be $\mathfrak{T}$ lower semicontinuous for all $y \in M$. If there exists $s_0 \in S$ such that*

$$M_{\#} = \overline{T_{s_0}(M)}^{\mathfrak{T}}$$

is $\mathfrak{T}$ compact and $T_s \circ T_t(x) = T_t \circ T_s(x)$ holds or all $s, t \in S$ and all $x \in M_{\#}$, then $P(S, M_{\#}) \neq \emptyset$. Moreover, $B_M(p)$ is nonempty and it carries a $\mathfrak{S}$ invariant probability measure.

Proof. Let us denote

$$S_{\#} = \{ss_0 : s \in S\} \cup \{s_0\}.$$

Clearly it is a semitopological subsemigroup of S (with the inherited topology) and the antirepresentation $\mathfrak{S}_{\#} = \{T_s : s \in S_{\#}\}$ is $\mathfrak{T}$ continuous. We notice that for every $s \in S$ one has

$$\overline{T_{ss_0}(M)}^{\mathfrak{T}} = \overline{T_{s_0}(T_s(M))}^{\mathfrak{T}} \subseteq \overline{T_{s_0}(M)}^{\mathfrak{T}} = M_{\#}.$$

This implies that the set $\overline{\{T_s(x) : s \in S_{\#}, x \in M\}}^{\mathfrak{T}}$ is $\mathfrak{T}$ compact and $\mathfrak{S}_{\#}$ invariant. We have already mentioned that commutative semigroups are amenable. Therefore, by Lemma 1(α), there exists an $\mathfrak{S}_{\#}$ invariant probability measure μ which is concentrated on $M_{\#}$. We have $\mu(T_{s_0}^{-1}(A)) = \mu(A)$ and $\mu(T_{ss_0}^{-1}(A)) = \mu(A)$ for all $s \in S$ and all $A \in \mathscr{B}_{\|\cdot\|, M}$. Now for arbitrary $s \in S$ and Borel $A \subseteq M$, we have

$$\begin{aligned}
\mu(T_s^{-1}(A)) &= \mu(T_s^{-1}(A) \cap M_{\#}) = \mu(T_{s_0}^{-1}(T_s^{-1}(A) \cap M_{\#})) \\
&= \mu(T_{s_0}^{-1}(T_s^{-1}(A)) \cap M_{\#}) = \mu(\{x \in M_{\#} : T_s \circ T_{s_0}(x) \in A\}) \\
&= \mu(\{x \in M_{\#} : T_{s_0} \circ T_s(x) \in A\}) = \mu(\{x \in M_{\#} : T_{ss_0}(x) \in A\}) \\
&= \mu(T_{ss_0}^{-1}(A)) = \mu(A).
\end{aligned}$$

Hence μ is $\mathfrak{S}$ invariant. This proves $\emptyset \neq P(S, M) = P(S_{\#}, M_{\#})$.

In order to show that there are $\mathfrak{S}$ invariant measures concentrated on $B_M(p)$, we first notice that $C_r \cap M_{\#}$ are nonempty, $\mathfrak{T}$ compact, and $\mathfrak{S}$ invariant for every $r > \delta(p, M)$. Therefore their intersections form a nonempty, $\mathfrak{T}$ compact, and $\mathfrak{S}$ invariant subset of $B_M(p)$. Now it is enough to repeat the arguments from the above. ∎

The next result is a two-fold generalization of Theorem 4.2 from [2]. We eliminate linearity assumptions imposed on T and I and also extend Al-Thagafi's results to more general (nonmonothetic) semigroups.

THEOREM 4. *Let M be a $\mathfrak{T}$ closed convex subset of a Banach space $(X, \|\cdot\|, \mathfrak{T})$ and $\mathfrak{S}=\{T_s: s\in S\}$ be a nonexpansive and $\mathfrak{T}$ continuous antirepresentation of S on M. Suppose that for some $s_0\in S$ the set $M_\sharp=\overline{T_{s0}(M)}^{\mathfrak{T}}$ is $\mathfrak{T}$ compact, norm separable, and $T_s\circ T_t(x)=T_t\circ T_s(x)$ holds for all $s, t\in S$ and all $x\in M_\sharp$. Then $F(S, B_M(p))\neq\emptyset$.*

Proof. It has already been noticed that $B_M(p)\neq\emptyset$ is convex, $\mathfrak{T}$ closed, and $\mathfrak{S}$ invariant. Let $\mathscr{K}$ be the family of all $\mathfrak{T}$ closed, convex nonempty subsets K of $B_M(p)$ which are $\mathfrak{S}$ invariant. It follows from the $\mathfrak{T}$ compactness of $M_\sharp\cap B_M(p)$ that Kuratowski–Zorn's Lemma is applicable and therefore there exists a minimal element K_0 in $\mathscr{K}$. In the same way as in the proof of Theorem 1 in [4], we obtain that $\emptyset\neq F(\mathscr{S}_\sharp, B_M(p))\ni x_0$. Now if $s\in S$ is arbitrary, then

$$T_s x_0 = T_s\circ T_{s0}x_0 = T_{s0}\circ T_s x_0 = T_{ss0}x_0 = x_0.$$

In particular $F(S, B_M(p)) = F(S_\sharp, B_M(p))\neq\emptyset$. ∎

The following result provides an explicit answer to the question raised in [2]. Here $\mathfrak{T}$ is the norm topology. Because the commutativity always guarantees amenability (even if the topology on $S_\sharp$ is discrete), we thus deal actually with a "discrete × norm" continuous antirepresentation on $M_\sharp$. In particular, all conditions of Theorem 4 are fulfilled. We have:

COROLLARY. *Let M be a closed, convex subset of a Banach space $(X, \|\cdot\|)$ containing the zero vector, $p\in X\backslash M$, and let $\mathfrak{S}=\{T_s: s\in S\}$ be a nonexpansive antirepresentation of a (discrete) semigroup S on $M\cup\{p\}$ such that $p\in F(S)$. If there exists $s_0\in S$ such that $M_\sharp=\overline{T_{s0}(M_p)}^{\|\cdot\|}$ is norm compact and $T_t\circ T_s(x)=T_s\circ T_t(x)$ holds for all $s, t\in S$ and $x\in M_\sharp$, then*

(i) *$B_M(p)\neq\emptyset$ is closed, convex and $\mathfrak{S}$ invariant,*

(ii) *$F(S, B_M(p))\neq\emptyset$.*

Proof. (i) is a part of our Theorem 2 and (ii) follows from Theorem 4. ∎

Linearity conditions imposed on I and T in [2] guarantee that the action of the semigroup generated by mappings I, T is affine. Hence it is weakly continuous. We show that the linearity condition imposed on T in Theorem 1(b) may be replaced by weak continuity. We have:

THEOREM 5. *Let* $(X, \|\cdot\|)$ *be a Banach space,* $0 \in M$ *be a convex and norm closed subset of* X, $p \in X \backslash M$, *and* $\mathfrak{S} = \{T_s : s \in S\}$ *be a weakly continuous, nonexpansive antirepresentation of a (discrete) semigroup* S *on* $M \cup \{p\}$ *such that* p *is a common fixed pont. Suppose that for some* $s_0 \in S$ *we have*

(a) T_{s_0} *is affine on* M_p,

(b) $\overline{T_{s_0}(M_p)}^w = M_\sharp$ *is weakly compact,*

(c) $T_s \circ T_t(x) = T_t \circ T_s(x)$ *holds for all* $s, t \in S$ *and* $x \in M_\sharp$.

Then

(i) $\varnothing \neq B_M(p)$ *is closed and convex,*

(ii) $T_s(B_M(p)) \subseteq B_M(p)$ *for every* $s \in S$,

(iii) $F(S, B_M(p)) \neq \varnothing$.

Proof. (i) can be proved in a similar way as in Theorem 2 and (ii) follows from nonexpansiveness. To prove (iii), we again introduce the semigroup $S_\sharp = \{ss_0 : s \in S\} \cup \{s_0\}$. The weakly compact set $M_\sharp$ is $\mathfrak{S}$-invariant and the corresponding antirepresentation $\mathfrak{S}_0$ restricted to $M_\sharp$ is commutative. The same properties are enjoyed by the $\mathfrak{S}$-invariant subset $K = \overline{T_{s_0}(B_M(p))}^w \subseteq M_\sharp$ which is convex as T_{s_0} is affine. It follows from Lemma 1 that the set K has a $\mathfrak{S}_\sharp$ common fixed point x_0. By an analogous argument as we have used in the proof of Theorem 3, we obtain

$$T_s x_0 = T_s \circ T_{s_0} x_0 = T_{s_0} \circ T_s x_0 = T_{ss_0} x_0 = x_0$$

for all $s \in S$. In particular $F(S, B_M(p)) = F(S_\sharp, B_M(p)) \neq \varnothing$. ∎

Remark 4. Finally we want to emphasize that in the last theorem we do not need to assume the norm separability of $M_\sharp$. A detailed explanation can be found in the last few paragraphs of [4]. It also follows from [4] that Theorem 5 has its "weak* version." However, in this case the norm separability of $M_\sharp$ has to be restored. It is still an open question (see [4] Problem) whether the norm separability condition is essential in Lemma 1 (γ) if $\mathfrak{T}$ is the weak* topology.

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